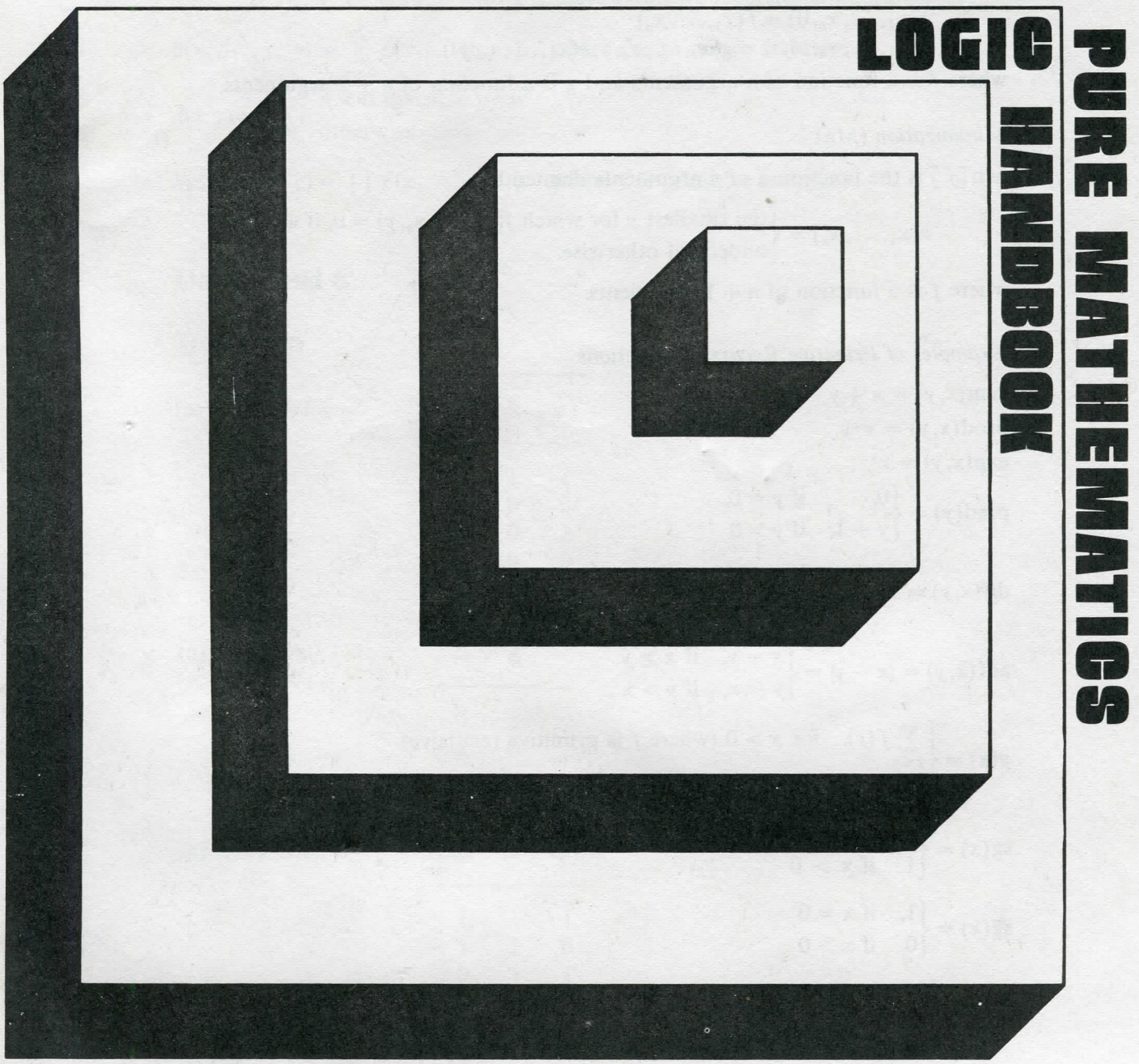
M335 L HB

THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M335 Studies in Pure Mathematics

You are strongly advised to take this Handbook into the examination with you as the exam paper may refer to it. However, the only annotation that will be permitted by the invigilator is the correction of any errors and underlining of passages, so take care not to write any of your own notes in this Handbook.

STUDIES IN PURE MATHEMATICS

LOGIC HANDBOOK



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Units 3 and 4

Recursive Functions

The *initial* functions are as follows:

The zero function z where $z(x) = 0$ for all x. The successor function s where $s(x) = x + 1$ for all x.
The identity functions id and id_i^n where

$$\mathrm{id}(x) = x \quad \text{for all } x$$

and $\quad \mathrm{id}_i^n(x_1, x_2, \ldots, x_n) = x_i \quad \text{for all } x_1, x_2, \ldots, x_n.$

Composition (Cn)

$\mathrm{Cn}[f, g_1, \ldots, g_m]$ is the function h of n arguments defined by

$$h(x_1, \ldots, x_n) = f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$$

where f is a function of m arguments and each g_i is a function of n arguments.

Primitive Recursion (Pr)

$\mathrm{Pr}[f, g]$ is the function h of $n + 1$ arguments defined by

$$h(x_1, \ldots, x_n, 0) = f(x_1, \ldots, x_n)$$
$$h(x_1, \ldots, x_n, s(y)) = g(x_1, \ldots, x_n, y, h(x_1, \ldots, x_n, y))$$

where f is a function of n arguments and g is a function of $n + 2$ arguments.

Minimization (Mn)

$\mathrm{Mn}[f]$ is the function h of n arguments defined by

$$h(x_1, \ldots, x_n) = \begin{cases} \text{the smallest } y \text{ for which } f(x_1, \ldots, x_n, y) = 0, \text{ if any,} \\ \text{undefined otherwise,} \end{cases}$$

where f is a function of $n + 1$ arguments.

Examples of Primitive Recursive Functions

$\mathrm{sum}(x, y) = x + y$

$\mathrm{prod}(x, y) = x \cdot y$

$\exp(x, y) = x^y$

$$\mathrm{pred}(y) = \begin{cases} 0, & \text{if } y = 0, \\ y - 1, & \text{if } y > 0 \end{cases}$$

$$\mathrm{dif}(x, y) = x \dot{-} y = \begin{cases} x - y, & \text{if } x \geq y \\ 0, & \text{if } x < y \end{cases}$$

$$\mathrm{adf}(x, y) = |x - y| = \begin{cases} x - y, & \text{if } x \geq y \\ y - x, & \text{if } y > x \end{cases}$$

$$g(x) = \begin{cases} \sum_{y<x} f(y), & \text{for } x > 0 \text{ (where } f \text{ is primitive recursive)} \\ 0, & \text{for } x = 0 \end{cases}$$

$$\mathrm{sg}(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$$

$$\overline{\mathrm{sg}}(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases}$$

$$e(x) = \begin{cases} 0, & \text{if } x \text{ is even} \\ 1, & \text{if } x \text{ is odd} \end{cases}$$

$$f(x,y) = \begin{cases} g_1(x,y), & \text{if } C_1 \\ \vdots & \vdots \\ g_n(x,y), & \text{if } C_n \end{cases}$$

where $g_1, \ldots, g_n$ are primitive recursive functions, where $C_1, \ldots, C_n$ are mutually exclusive, collectively exhaustive conditions on pairs (x,y) of natural numbers, and where the conditions are also primitive recursive. (A condition C_i is primitive recursive if its characteristic function c_i defined by

$$c_i(x,y) = \begin{cases} 1, & \text{if } C_i \text{ holds for } (x,y), \\ 0, & \text{if } C_i \text{ does not hold for } (x,y), \end{cases}$$

is primitive recursive.)

$$\mathrm{Eq}(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$$

$$f(x,y) = \begin{cases} g_1(x,y), & \text{if } C_1 \\ \vdots & \vdots \\ g_n(x,y), & \text{if } C_n \\ g_{n+1}(x,y), & \text{otherwise} \end{cases}$$

where $g_1, \ldots, g_{n+1}$ are primitive recursive functions and where $C_1, \ldots, C_n$ are mutually exclusive primitive recursive conditions.

$$h(x_1, \ldots, x_n, y) = \sum_{z=0}^{y} f(x_1, \ldots, x_n, z), \quad \text{where } f \text{ is primitive recursive.}$$

$$d(x,y) = \begin{cases} 1, & \text{if } x \text{ is divisible by } y \\ 0, & \text{otherwise} \end{cases}$$

$$h(x_1, \ldots, x_n, y) = \prod_{z=0}^{y} f(x_1, \ldots, x_n, z), \quad \text{where } f \text{ is primitive recursive.}$$

Units 5 and 6

Truth Tables

(*a*) *Conjunction*

$(\phi$	&	$\psi)$
1	1	1
1	0	0
0	0	1
0	0	0

(*b*) *Negation*

–	ϕ
0	1
1	0

(*c*) *Disjunction*

$(\phi$	$\vee$	$\psi)$
1	1	1
1	1	0
0	1	1
0	0	0

(*d*) *Implication*

$(\phi$	$\longrightarrow$	$\psi)$
1	1	1
1	0	0
0	1	1
0	1	0

(*e*) *Bi-implication*

ϕ	$\longleftrightarrow$	ψ
1	1	1
1	0	0
0	0	1
0	1	0

Rules of Proof

Tautology Rule (*Taut*)

If the formulas $\phi_1,\ldots,\phi_k$ occur in a formal proof and the formula ψ is a tautological consequence of $\phi_1,\ldots,\phi_k$, then on any line subsequent to the lines on which $\phi_1,\ldots,\phi_k$ appear we may introduce the formula ψ. The assumptions on which ψ depends are defined to be all those on which $\phi_1,\ldots,\phi_k$ depend.

Conditional Proof (*CP*)

If a formula ψ occurs on a line in a formal proof, and a formula ϕ occurs amongst the assumptions in force on that line, then on any subsequent line we may introduce the formula $(\phi \longrightarrow \psi)$. The assumptions on which $(\phi \longrightarrow \psi)$ depends are all those on which ψ depends except for ϕ.

Universal Quantifier Elimination Rule (*UE*)

If the formula $\forall v\,\phi$ occurs on a certain line in a formal proof and τ is a term which may be freely substituted for the variable v in ϕ, then on any subsequent line we may derive the formula $\phi(\tau/v)$, and this formula will depend on the same assumptions as did $\forall v\,\phi$.

(A term τ may be *freely substituted for the variable x in the formula* ϕ if no variable in τ becomes bound when free occurrences of x in ϕ are replaced by occurrences of τ.)

Universal Quantifier Introduction Rule (*UI*)

If the formula ϕ occurs on a line of a formal proof and the variable v does not occur free in any of the assumptions in force on this line, then on any subsequent line we may derive $\forall v\,\phi$, and this formula will depend on the same assumptions as did ϕ.

Existential Quantifier Introduction Rule (*EI*)

If τ is a term which may be freely substituted for the variable v in the formula ϕ, and the formula $\phi(\tau/v)$ occurs on a line of a formal proof, then on any subsequent line we may derive the formula $\exists v\,\phi$, and this formula will depend on the same assumptions as did $\phi(\tau/v)$.

Existential Hypothesis Rule (EH)

Let ψ be a formula which contains no free occurrences of the variable v. If on a certain line of a formal proof we have derived the formula ψ from the assumption ϕ and possibly other assumptions in which v does not occur free (though v may occur free in ϕ), then on any subsequent line we may derive ψ from $\exists v\,\phi$ and these other assumptions.

Identity Introduction Rule (II)

For each term τ the formula $\tau = \tau$ may be written down on any line of a proof and does not depend on any assumptions.

Substitution Rule (SR)

Suppose that each of the terms τ_1 and τ_2 may be freely substituted for the variable v in the formula ϕ. Let ϕ_1 be a formula which results when some (but not necessarily all) of the free occurrences of v in ϕ are replaced by τ_1, and let ϕ_2 be the formula that results when these same occurrences are replaced by τ_2. Then if on two lines of a formal proof the formulas $\tau_1 = \tau_2$ and ϕ_1 occur, on any subsequent line we may derive ϕ_2, and ϕ_2 will depend on all the assumptions in force on the lines on which the formulas $\tau_1 = \tau_2$ and ϕ_1 occur.

The Correctness Theorem

If $\phi_1,\ldots,\phi_k \vdash \psi$, then ψ is a logical consequence of the formulas $\phi_1,\ldots,\phi_k$.

The Adequacy Theorem

If ψ is a logical consequence of the formulas $\phi_1,\ldots,\phi_k$, then $\phi_1,\ldots,\phi_k \vdash \psi$.

Unit 7

Axioms of Q

Q1 $\forall x \forall y\ (x' = y' \longrightarrow x = y)$
Q2 $\forall x\ -\mathbf{0} = x'$
Q3 $\forall x\ (-x = \mathbf{0} \longrightarrow \exists y\ x = y')$
Q4 $\forall x\ (x + \mathbf{0}) = x$
Q5 $\forall x \forall y\ (x + y') = (x + y)'$
Q6 $\forall x\ (x \cdot \mathbf{0}) = \mathbf{0}$
Q7 $\forall x \forall y\ (x \cdot y') = ((x \cdot y) + x)$

*Interpretation N**

Domain $N \cup \{\omega\}$ with functions

x	0	1	$\cdots$	n	$\cdots$	ω
x'	1	2	$\cdots$	$n+1$	$\cdots$	ω

				y			
$x + y$	0	1	2	$\cdots$	n	$\cdots$	ω
0	0	1	2	$\cdots$	n	$\cdots$	ω
1	1	2	3	$\cdots$	$1+n$	$\cdots$	ω
2	2	3	4	$\cdots$	$2+n$	$\cdots$	ω
x $\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
m	m	$m+1$	$m+2$	$\cdots$	$m+n$	$\cdots$	ω
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
ω	ω	ω	ω	$\cdots$	ω	$\cdots$	ω

$x \cdot y$	y						
	0	1	2	$\cdots$	n	$\cdots$	ω
0	0	0	0	$\cdots$	0	$\cdots$	0
1	0	1	2	$\cdots$	n	$\cdots$	ω
2	0	2	4	$\cdots$	$2n$	$\cdots$	ω
x $\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
m	0	m	$2m$	$\cdots$	mn	$\cdots$	ω
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
ω	0	ω	ω	$\cdots$	ω	$\cdots$	ω

*Interpretation N***

Domain $N \cup \{\alpha, \beta\}$ where $\alpha \neq \beta$, with functions

x	0	1	2	$\cdots$	n	$\cdots$	α	β
x'	1	2	3	$\cdots$	$n+1$	$\cdots$	α	β

$x + y$	y							
	0	1	2	$\cdots$	n	$\cdots$	α	β
0	0	1	2	$\cdots$	n	$\cdots$	β	α
1	1	2	3	$\cdots$	$n+1$	$\cdots$	β	α
2	2	3	4	$\cdots$	$n+2$	$\cdots$	β	α
x $\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
m	m	$m+1$	$m+2$	$\cdots$	$m+n$	$\cdots$	β	α
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
α	α	α	α	$\cdots$	α	$\cdots$	β	α
β	β	β	β	$\cdots$	β	$\cdots$	β	α

$x \cdot y$	y							
	0	1	2	$\cdots$	n	$\cdots$	α	β
0	0	0	0	$\cdots$	0	$\cdots$	α	β
1	0	1	2	$\cdots$	n	$\cdots$	α	β
2	0	2	4	$\cdots$	$2n$	$\cdots$	α	β
x $\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
m	0	m	$2m$	$\cdots$	mn	$\cdots$	α	β
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
α	0	β	β	$\cdots$	β	$\cdots$	β	β
β	0	α	α	$\cdots$	α	$\cdots$	α	α

A function f of n variables is *representable in* T if there is a formula ϕ in which no variables other than $x_1, \ldots, x_n, x_{n+1}$ occur free such that if $p_1, \ldots, p_n$ and j are natural numbers with $f(p_1, \ldots, p_n) = j$ then

(i) $\vdash_T \phi(\mathbf{p_1}, \ldots, \mathbf{p_n}, \mathbf{j})$

and (ii) $\vdash_T \forall x_{n+1}(\phi(\mathbf{p_1}, \ldots, \mathbf{p_n}, x_{n+1}) \longrightarrow x_{n+1} = \mathbf{j})$.

The Representability Theorem

Every recursive function is representable in Q.

Unit 8

Gödel Numbers

Symbol	(	)	&	$\vee$	$\longrightarrow$	$\longleftrightarrow$	$-$	$\exists$	$\forall$	**0**
gn	1	2	3	39	399	3999	39999	4	49	6

Symbol	$'$	$+$	$\cdot$	$=$	x	y	z
gn	68	688	6889	788	5	59	599

Axioms of Z (Elementary Peano Arithmetic)

Q1 to *Q7* of the system *Q*.

All sentences obtained from formulas of the form

$$([A(\mathbf{0}) \;\&\; \forall x\,(A(x) \longrightarrow A(x'))] \longrightarrow \forall x\, A(x))$$

by prefixing universal quantifiers.

Results

(*1*) *Gödel's Diagonal Lemma* (**BJ** *Lemma 2, page 173*)

Let T be an axiom system in which the function diag is representable. Then given any formula $B(y)$ in one free variable y, we can find a corresponding sentence G such that

$$\vdash_T (G \longleftrightarrow B(\ulcorner G \urcorner)).$$

(*2*) **BJ** *Lemma 3, page 174*

If T is a consistent extension of Q, then the set of gödel numbers of theorems of T is not definable in T.

(*3*) **BJ** *Theorem 1, page 175*

No consistent extension of Q is decidable.

(*4*) **BJ** *Lemma 4, page 175*

Q is not decidable.

(*5*) **BJ** *Theorem 3, page 176*

Arithmetic is not decidable.

(*6*) *Theorem 1 of Unit 8, Section 8.4*

Let T be an axiom system with a decidable set of axioms. Then the set of gödel numbers of the theorems of T is a recursively enumerable set.

(*7*) *Theorem 2 of Unit 8, Section 8.4*

The theorems of a complete and consistent axiomatizable theory form a decidable set.

(*8*) *Gödel's First Incompleteness Theorem* (**BJ** *Theorem 6, page 179*)

There is no complete consistent axiomatizable extension of Q.

(*9*) *Gödel's Second Incompleteness Theorem* (*Unit 8, Section 8.6*)

If Z is consistent then $-\mathrm{Prov}(\ulcorner \mathbf{0} = \mathbf{1} \urcorner)$ is not a theorem of Z.

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